

Linear mappings of local preserving-majorization on matrix algebras¹

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Abstract

Let $\mathbf{M}_{n \times n}$ be the algebra of all $n \times n$ matrices. For $x, y \in \mathbf{R}^n$ it is said that x is majorized by y if there is a double stochastic matrix $A \in \mathbf{M}_{n \times n}$ such that $x = Ay$ (denoted by $x \prec y$). Let $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ in \mathbf{R}^n , then $x \prec y$ if and only if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$. Suppose that Φ is a linear mapping from \mathbf{R}^n into \mathbf{R}^n , which is said to be strictly isotone if $\Phi(x) \prec \Phi(y)$ whenever $x \prec y$. We say that an element $\alpha \in \mathbf{R}^n$ is a strictly all-isotone point if every strictly isotone φ at α (i.e. $\Phi(\alpha) \prec \Phi(y)$ whenever $x \in \mathbf{R}^n$ with $\alpha \prec x$, and $\Phi(x) \prec \Phi(\alpha)$ whenever $x \in \mathbf{R}^n$ with $x \prec \alpha$) is a strictly isotone. In this paper we show that every $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{R}^n$ with $\alpha_1 > \alpha_2 > \dots > \alpha_n$ is a strictly all-isotone point.

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1. Introduction and preliminaries

Let \mathbf{R}^n be real n -dimensional Euclidean spaces. $L(\mathbf{R}^n, \mathbf{R}^m)$ stands for the set of all linear mappings from \mathbf{R}^n into \mathbf{R}^m , and abbreviate $L(\mathbf{R}^n, \mathbf{R}^n)$ to $L(\mathbf{R}^n)$. We denote by $\mathbf{M}_{n \times m}$ the set of all $n \times m$ matrices. The cardinal number of a set A is denoted by $|A|$. \mathbf{N} is the set of non-negative integers.

For $x, y \in \mathbf{R}^n$ it is said that x is majorized by y if there is a double stochastic matrix $A \in \mathbf{M}_{n \times n}$ such that $x = Ay$. Let us write $x \sim y$ if $x \prec y$ and $y \prec x$. If we write $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$, then $x \prec y$ if and only if $\sum_{i=1}^k x_i \leq \sum_{i=1}^k y_i$ ($i = 1, 2, \dots, n-1$) and $\sum_{i=1}^n x_i = \sum_{i=1}^n y_i$ (see [9, Theorem 11.2]). A linear mapping $\Phi \in L(\mathbf{R}^n, \mathbf{R}^m)$ is said to be strictly isotone if $\Phi(x) \prec \Phi(y)$ whenever $x \prec y$; Φ is said to be a strictly isotone at α if $\Phi(P\alpha) \prec \Phi(x)$ whenever $x \in \mathbf{R}^n$ with $\alpha \prec x$, and $\Phi(x) \prec \Phi(\alpha)$ whenever $x \in \mathbf{R}^n$ with $x \prec \alpha$. A vector $\alpha \in \mathbf{R}^n$ is said to be strictly all-isotone point if every strictly isotone at α is a strictly isotone.

Some of our notations and symbols are explained as the following.

\mathbf{R}^n : the set of all $n \times 1$ real column vectors.

\mathbf{R}_n : the set of all $n \times 1$ real row vectors.

\mathbf{P}_n : the set of all $n \times n$ permutation matrices.

\mathbf{S}_n : the set of all $n \times n$ double stochastic matrices.

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With the development of majorization problem, preserving majorization have attracted much attention of mathematicians as an active subject of research in algebras. We describe some of the results related to ours. T. Anto [1] obtained the following interesting result.

Theorem 1.1 A linear mapping $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ satisfies $\Phi(x) \prec \Phi(y)$ whenever $x \prec y$ if and only if one of the following holds:

- 1) $\Phi(x) = (trx)a$ for some $a \in \mathbf{R}^n$;
- 2) $\Phi(x) = \alpha P(x) + \beta(trx)e$ for some $\alpha, \beta \in \mathbf{R}$ and $P \in \mathbf{P}_n$.

Hereafter the linear preservers of majorization are fully characterized on the algebra of all $n \times n$ matrices by C.K. Li and E. Poon in [7]. A. Armandnejad, F. Akbarzadeh and Z. Mohammadi [2] obtained many results of preserving row and column-majorization on $\mathbf{M}_{n \times m}$. On the other hand, over the past few years a considerable attention has been paid to the question of determining derivations (multiplicative mappings) through one point derivable (multiplicative) mappings (see [3,5,6,8,10-12] and references therein). The purpose of this paper is to characterize those linear mappings of preserving-majorization at one fixing point.

This paper is organized as follows: in section 2, first we will introduce some notations of preserving majorization at one point, then we will give the main theorem in this paper and two lemmas which requires them in the proof of the main theorem. In section 3 we will give the proof of the main theorem.

2. Two lemmas and the main theorem

Given two real vectors $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ in \mathbf{R}^n , let $x^\cdot = (x_1^\cdot, x_2^\cdot, \dots, x_n^\cdot)^T$ and $x_\cdot = (x_{\cdot 1}, x_{\cdot 2}, \dots, x_{\cdot n})^T$ denote x and y decreasing rearrangement and increasing rearrangement, respectively, i.e. $x_1^\cdot \geq x_2^\cdot \geq \dots \geq x_n^\cdot$ and $x_{\cdot 1} \leq x_{\cdot 2} \leq \dots \leq x_{\cdot n}$. The trace of x is $tr(x) = \sum_{k=1}^n x_k$. The set $\{P \in \mathbf{P}_n : (P^T x)^T y^\cdot = M(x, y)\}$ and $\{P \in \mathbf{P}_n : (P^T x)^T y_\cdot = m(x, y)\}$ are denoted by $I_M(x, y)$ and $I_m(x, y)$, respectively, where $M(x, y) = (x^\cdot)^T y^\cdot$ and $m(x, y) = (x_\cdot)^T y_\cdot$. We may identify an $n \times n$ matrix A_Φ as a linear mapping Φ on \mathbf{R}^n .

Definition 2.1 Let $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a linear mapping.

- 1) Φ is said to be a strictly left-isotone at $\alpha \in \mathbf{R}^n$ if $\Phi(y) \prec \Phi(P\alpha)$ whenever $P \in \mathbf{P}_n$ and $y \in \mathbf{R}^n$ with $y \prec \alpha$.
- 2) Φ is said to be a strictly right-isotone at $\alpha \in \mathbf{R}^n$ if $\Phi(P\alpha) \prec \Phi(y)$ whenever $P \in \mathbf{P}_n$ and $y \in \mathbf{R}^n$ with $\alpha \prec y$.
- 3) Φ is said to be a strictly isotone at $\alpha \in \mathbf{R}^n$ if $\Phi(y) \prec \Phi(\alpha)$ whenever $y \in \mathbf{R}^n$ with $y \prec \alpha$, and $\Phi(\alpha) \prec \Phi(y)$ whenever $y \in \mathbf{R}^n$ with $\alpha \prec y$.
- 4) Φ is said to be a strictly preserving equivalence at $\alpha \in \mathbf{R}^n$ if $\Phi(y) \sim \Phi(\alpha)$ whenever $y \in \mathbf{R}^n$ with $y \sim \alpha$.

In the rest part of this paper, we always assume that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)^T$ with $\alpha_1 > \alpha_2 > \dots > \alpha_n$. The following theorem is our main result.

Theorem 2.2 Let $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ be a linear mapping. Then the following statements are mutually equivalent:

- 1) Φ is a strictly left-isotone at $\alpha \in \mathbf{R}^n$.
- 2) Φ is a strictly right-isotone at $\alpha \in \mathbf{R}^n$.
- 3) Φ is a strictly isotone at $\alpha \in \mathbf{R}^n$.
- 4) Φ is a strictly preserving equivalence at $\alpha \in \mathbf{R}^n$.
- 5) Φ is a strictly isotone.

The following most famous inequality for vectors is due to G.H. Hardy, J.E. Littlewood, G. Polya in [4, Theorem 368].

Lemma 2.3 Let $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ be two vectors in \mathbf{R}^n , then

(1) $m(x, y) = (x^*)^T y^* \leq (Px)^T y \leq (x^*)^T y^* = M(x, y), \forall P \in \mathbf{P}_n$.

(2) If $y^* = (y_1, y_2, \dots, y_n)$ with $y_1 > y_2 > \dots > y_n$, then $(Px)^T y^* = M(x, y)$ if and only if $Px = x^*$.

(3) If $y^* = (y_1, y_2, \dots, y_n)$ with $y_1 > y_2 > \dots > y_n$, then $(Px)^T y^* = m(x, y)$ if and only if $Px = x^*$.

Proof. (1) The inequality for vectors is due to G.H. Hardy, J.E. Littlewood, G. Polya in [4, Theorem 368].

(2) The fact that the condition $Px = x^*$ is sufficient by the inequality for vectors in (1). We only need to prove the necessity of the statement. In fact, if $Px \neq x^*$, we write $Px = (x_{P(1)}, x_{P(2)}, \dots, x_{P(n)})$, then there are $1 \leq m < k \leq n$ such that $x_{P(k)} < x_{P(m)}$. Since $(x_{P(m)} - x_{P(k)})(y_k - y_m) > 0$, we have $x_{P(k)}y_k + x_{P(m)}y_m < x_{P(m)}y_k + x_{P(k)}y_m$. It follows that

$$\begin{aligned} (Px)^T y^* &= \sum_{i=1}^n x_{P(i)} y_i = \sum_{i \neq k, m} x_{P(i)} y_i + x_{P(k)} y_k + x_{P(m)} y_m \\ &< \sum_{i \neq k, m} x_{P(i)} y_i + x_{P(m)} y_k + x_{P(k)} y_m \leq M(x, y). \end{aligned}$$

Hence $(Px)^T y^* \neq M(x, y)$.

(3) The statement can be proved by imitating the proof in (2).

This completes the proof of the lemma. \square

Lemma 2.4 Let $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T$ be two vectors in \mathbf{R}^n and $y_1 > y_2 > \dots > y_n$. If there are k unequal components at least in all entries of x , then $|I_M(x, y^*)| \leq (n - k + 1)!$ and $|I_m(x, y^*)| \leq (n - k + 1)!$.

Proof. For $P \in I_M(x, y^*)$, i.e. $(P^T x)^T y^* = M(x, y) = (x^*)^T y^*$, it implies from Lemma 2.3 that

$$P^T x = (x_{P^T(1)}, x_{P^T(2)}, \dots, x_{P^T(n)})^T = x^*,$$

i.e. $x_{P^T(1)} \geq x_{P^T(2)} \geq \dots \geq x_{P^T(n)}$. Note that there are k unequal components at least in all entries of x , then we have

$$\begin{aligned} x_{P^T(1)} &= x_{P^T(2)} = \dots = x_{P^T(l_1)} \\ &> x_{P^T(l_1+1)} = x_{P^T(l_1+2)} = \dots = x_{P^T(l_2)} \\ &> x_{P^T(l_2+1)} = x_{P^T(l_2+2)} = \dots = x_{P^T(l_3)} \\ &\dots \\ &> x_{P^T(l_{k-2}+1)} = x_{P^T(l_{k-2}+2)} = \dots = x_{P^T(l_{k-1})} \\ &> x_{P^T(l_{k-1}+1)} \geq x_{P^T(l_{k-1}+2)} \geq \dots \geq x_{P^T(n)}. \end{aligned}$$

It is easy to see that $|I_M(x, y)| \leq l_1!(l_2 - l_1)! \dots (n - l_{k-1})! \leq (n - k + 1)!$.

Similarly, we can prove that $|I_m(x, y^*)| \leq (n - k + 1)!$. This completes the proof. \square

Note. Obviously $I_M(x, y^*) = \{P : (P^T x)^T y^* = M(x, y)\} = \{P : x^T P y^* = M(x, y)\}$ and $I_m(x, y^*) = \{P : (P^T x)^T y^* = m(x, y)\} = \{P : x^T P y^* = m(x, y)\}$.

3. The proof of the main theorem

The proof of Theorem 2.2 From Definition 2.1 we can easily prove that "1) \Rightarrow 4)", "2) \Rightarrow 4" and "3) \Rightarrow 4". It is easy to see that 5) implies 1)-3). So we only need to show that 4) implies 5).

Suppose that Φ is a linear mapping from \mathbf{R}^n into itself which satisfies strictly preserving equivalence at $\alpha \in \mathbf{R}^n$, we write the $n \times n$ matrix $A_\Phi = (a_{ij})$, and abbreviate A_Φ to A . Since there is a real number $\lambda \in \mathbf{R}$ such that $A + \lambda J = (b_{ij})$ with $b_{ij} > 0$ (where every entries of the $n \times n$ matrix J is 1). It suffices to prove the theorem in the case of the matrix $A = (a_{ij})$ with $a_{ij} > 0$. We write $A = (a_1/a_2/\dots/a_n)$ ($A = (a^1/a^2/\dots/a^n)$), where a_j (a^j) is the j th row (column) of A .

For any $P \in \mathbf{P}_n$, it follows from $P\alpha \sim \alpha$ that $A(P\alpha) \sim A(\alpha)$. Thus we may write

$$(A\alpha)^\cdot = (M_1, M_2, \dots, M_n)^T,$$

and

$$\begin{aligned} AP\alpha &= (a_1P\alpha, a_2P\alpha, \dots, a_nP\alpha)^T \\ &= (M_{k_1}, M_{k_2}, \dots, M_{k_n})^T. \end{aligned}$$

Then there is an a_l at least such that $a_lP\alpha = M_1 = M(a_l, \alpha)$, i.e, $P \in I_M(a_l, \alpha)$.

We shall organize the proof of the theorem in a series of claims as follow.

Claim 1. First, we show that $tr(a^s) = tr(a^t)$ for every $1 \leq s, t \leq n$.

Taking two permutation matrices $P, Q \in \mathbf{P}_n$, then $AP\alpha \sim A\alpha \sim AQ\alpha$ by $P\alpha \sim \alpha \sim Q\alpha$. In particular

$$\sum_{l=1}^n \sum_{k=1}^n a_{lk} \alpha_{P(k)} = \sum_{l=1}^n \sum_{k=1}^n a_{lk} P\alpha = \sum_{k=1}^n M_k.$$

and

$$\sum_{l=1}^n \sum_{k=1}^n a_{lk} \alpha_{Q(k)} = \sum_{l=1}^n \sum_{k=1}^n a_{lk} Q\alpha = \sum_{k=1}^n M_k.$$

For any two nature numbers $1 \leq s, t \leq n$ with $t \neq s$, we take a permutation matrix $P \in \mathbf{P}_n$ such that $P(e_s) = e_t$, $P(e_t) = e_s$ and $P(e_k) = e_k$ whenever $k \neq s, t$, simultaneously we take $Q = I$ (where I is the unit matrix in \mathbf{P}_n). Thus we have $\sum_{l=1}^n (a_{ls} \alpha_t + a_{lt} \alpha_s) = \sum_{l=1}^n (a_{ls} \alpha_s + a_{lt} \alpha_t)$, i.e. $\sum_{l=1}^n a_{ls} (\alpha_t - \alpha_s) = \sum_{l=1}^n a_{lt} (\alpha_t - \alpha_s)$. Hence $tr(a^s) = tr(a^t)$.

Claim 2. Suppose that every row a_i of A contains two unequal components at least in all entries of a_i .

$$\text{We show that } AR = \begin{pmatrix} \gamma & \lambda & \dots & \lambda \\ \lambda & \gamma & \dots & \lambda \\ \dots & \dots & \dots & \dots \\ \lambda & \lambda & \dots & \gamma \end{pmatrix} \text{ for some } R \in \mathbf{P}_n \text{ and } \lambda, \gamma \in \mathbf{R} \text{ with } \lambda \neq \gamma.$$

Suppose that $n = 2$. Then we may write $A = \begin{pmatrix} \lambda_1 & \gamma_1 \\ \lambda_2 & \gamma_2 \end{pmatrix}$ and $\lambda_1 + \lambda_2 = \gamma_1 + \gamma_2$ by Claim

1. Without loss of generality, we assume that $\gamma_1 > \lambda_1$. Since $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in \mathbf{P}_2$ and $AP\alpha \sim A\alpha$,

$\begin{pmatrix} \gamma_1 & \lambda_1 \\ \gamma_2 & \lambda_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} \sim \begin{pmatrix} \lambda_1 & \gamma_1 \\ \lambda_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$. It follows that $\gamma_1 \alpha_1 + \lambda_1 \alpha_2 = \gamma_2 \alpha_2 + \lambda_2 \alpha_1 = M_1$ and $\gamma_2 \alpha_1 + \lambda_2 \alpha_2 = \gamma_1 \alpha_2 + \lambda_1 \alpha_1 = M_2$. Simple computation we obtain $\gamma_2 = \lambda_1$, and $\gamma_1 = \lambda_2$ by $\lambda_1 + \lambda_2 = \gamma_1 + \gamma_2$. Hence $A = \begin{pmatrix} \gamma_2 & \lambda_2 \\ \lambda_2 & \gamma_2 \end{pmatrix}$.

Suppose that $n \geq 3$. We divided the proof into four steps.

Step 1. We claim that every row a_i of A contains only two unequal components in all entries of a_i .

If the claim is not true, it follows that there is some row a_m of A such that it contains at least three unequal components in all entries of a_m . By Lemmma 2.4, $|I_M(a_m, \alpha)| \leq (n-2)!$, and $|I_M(a_k, \alpha)| \leq (n-1)!$ whenever $k \neq m$. Thus we have

$$\sum_{k=1}^n |I_M(a_k, \alpha)| \leq (n-2)! + (n-1)(n-1)! < n!.$$

On the other hand, for every $P \in \mathbf{P}_n$, we have $AP\alpha \sim A\alpha$. Thus there is some a_k such that $a_k P\alpha = M(a_k, \alpha)$, i.e. $P \in I_M(a_k, \alpha)$. It follows that

$$\sum_{k=1}^n |I_M(a_k, \alpha)| \geq |\mathbf{P}_n| \geq n!,$$

which is a contradiction. Hence the claim holds.

Step 2. We claim that every components of a_i is equal to same real number only except one, $\sum_{k=1}^n |I_M(a_l, \alpha)| = n!$ and $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$ for $l \neq k$.

If the claim is not true, then there are an a_{l_0} , $P \in \mathbf{P}_n$ and $2 \leq k \leq n-2$ such that $a_{l_0} = (a, \dots, a, b, \dots, b)P$, where a and b appears k times and $n-k$ times in components of a_{l_0} , respectively. By Lemma 2.3, we have $|I_M(a_{l_0}, \alpha)| \leq k!(n-k)! < (n-1)!$. Note that $|I_M(a_l, \alpha)| \leq (n-1)!$ whenever $l \neq l_0$, thus we obtain $\sum_{l=1}^n |I_M(a_l, \alpha)| < n!$. On the other hand, for every $P \in \mathbf{P}_n$, we have $AP\alpha \sim A\alpha$. Thus there is some a_k such that $a_k P\alpha = M(a_k, \alpha)$, i.e. $P \in I_M(a_k, \alpha)$. It follows that

$$\sum_{k=1}^n |I_M(a_k, \alpha)| \geq |\mathbf{P}_n| \geq n!,$$

which is a contradiction. Hence

$$a_l = (\lambda_l, \dots, \lambda_l, \gamma_l, \lambda_l, \dots, \lambda_l)$$

with $\lambda_l \neq \gamma_l$, ($l = 1, 2, \dots, n$).

It is easy to see from Lemma 2.3 and the form of the above vector that $|I_M(a_l, \alpha)| = (n-1)!$. So $\sum_{k=1}^n |I_M(a_l, \alpha)| = n!$. Since $|\mathbf{P}_n| = n!$, we have $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$ for $l \neq k$.

Step 3. We claim that every column a^s of A contains two γ_t 's at most in its components.

If the claim is not true, then there are three rows a_l, a_k and a_m of A such that

$$\begin{aligned} a_l &= (\lambda_l, \dots, \lambda_l, \gamma_l, \lambda_l, \dots, \lambda_l), \\ a_k &= (\lambda_k, \dots, \lambda_k, \gamma_k, \lambda_k, \dots, \lambda_k), \\ a_m &= (\lambda_m, \dots, \lambda_m, \gamma_m, \lambda_m, \dots, \lambda_m), \end{aligned}$$

where the j th components of a_l, a_k and a_m are γ_l, γ_k and γ_m , respectively. There are several possibilities.

Case 1. Suppose that $\gamma_l > \lambda_l$ and $\gamma_k > \lambda_k$. There are $P \in \mathbf{P}_n$ such that $a_l P\alpha = M(a_l, \alpha)$, i.e. $P \in I_M(a_l, \alpha)$. Obviously $P \in I_M(a_k, \alpha)$, this is a contradiction with $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$.

Case 2. Suppose that $\gamma_l < \lambda_l$ and $\gamma_k < \lambda_k$. There are $P \in \mathbf{P}_n$ such that $a_l P\alpha = M(a_l, \alpha)$, i.e. $P \in I_M(a_l, \alpha)$. Obviously $P \in I_M(a_k, \alpha)$, this is a contradiction with $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$.

Case 3. Suppose that $\gamma_m < \lambda_m$ and $\gamma_k < \lambda_k$. There are $P \in \mathbf{P}_n$ such that $a_m P\alpha = M(a_m, \alpha)$, i.e. $P \in I_M(a_m, \alpha)$. Obviously $P \in I_M(a_k, \alpha)$, this is a contradiction with $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$.

The other cases can beget the same contradiction by imitating the above Case 1, Case 2 and Case 3. Thus every column a^k of A contains two γ_i 's at most in its components.

Step 4. We claim that every column a^s of A includes only one γ_t 's in its components.

If the claim is not true, then there are two rows a_k and a_m of A such that

$$\begin{aligned} a_k &= (\lambda_k, \dots, \lambda_k, \gamma_k, \lambda_k, \dots, \lambda_k), \\ a_m &= (\lambda_m, \dots, \lambda_m, \gamma_m, \lambda_m, \dots, \lambda_m), \end{aligned}$$

where the j th components of a_k and a_m are γ_k and γ_m , respectively. At this time there must be some column a^l of A such that $a^l = (\lambda_1, \lambda_2, \dots, \lambda_n)^T$ and $\text{tr}(a^l) = \sum_{i=1}^n \lambda_i$.

There are several possibilities.

Case 1. Suppose that there is some column a^j of A such that a^j includes only one γ_t 's in its components. By Claim 1, we have $\gamma_t + \sum_{i \neq t} \lambda_i = \text{tr}(a^j) = \text{tr}(a^l) = \sum_{i=1}^n \lambda_i$, i.e. $\gamma_t = \lambda_t$, this is a contradiction with $\gamma_t \neq \lambda_t$.

Case 2. Suppose that every column a^j of A includes two γ_t 's or not in its components. In this case, n must be an even, which implies $n \geq 4$. Without loss of generality, we write

$$A = \begin{pmatrix} \gamma_1 & \lambda_1 & \lambda_1 & \cdots & \lambda_1 \\ \gamma_2 & \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \lambda_3 & \gamma_3 & \lambda_3 & \cdots & \lambda_3 \\ \lambda_4 & \gamma_4 & \lambda_4 & \cdots & \lambda_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \lambda_n & \lambda_n & \cdots & \cdots & \cdots \end{pmatrix}.$$

Since $\text{tr}(a^1) = \text{tr}(a^2) = \sum_{i=1}^n \lambda_i$, $\gamma_1 + \gamma_2 = \lambda_1 + \lambda_2$ and $\gamma_3 + \gamma_4 = \lambda_3 + \lambda_4$. Thus there are $0 \neq c, d \in \mathbf{R}$ such that $\gamma_1 = \lambda_1 + c$, $\gamma_2 = \lambda_2 - c$, $\gamma_3 = \lambda_3 + d$ and $\gamma_4 = \lambda_4 - d$.

$$\text{Take } P_0 = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \in \mathbf{P}_n. \text{ Then}$$

a) Suppose that $c > 0$ and $d > 0$. Then $P_0 \in I_M(a_2, \alpha) \cap I_M(a_3, \alpha)$ by Lemma 2.3, which is a contradiction with $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$ for $l \neq k$.

b) Suppose that $c > 0$ and $d < 0$. Then $P_0 \in I_M(a_2, \alpha) \cap I_M(a_4, \alpha)$ by Lemma 2.3, which is a contradiction with $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$ for $l \neq k$.

c) Suppose that $c < 0$ and $d > 0$. Then $P_0 \in I_M(a_1, \alpha) \cap I_M(a_3, \alpha)$ by Lemma 2.3, which is a contradiction with $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$ for $l \neq k$.

d) Suppose that $c < 0$ and $d < 0$. Then $P_0 \in I_M(a_1, \alpha) \cap I_M(a_4, \alpha)$ by Lemma 2.3, which is a contradiction with $I_M(a_l, \alpha) \cap I_M(a_k, \alpha) = \emptyset$ for $l \neq k$.

Hence every column a^s of A includes only one γ_t 's in its components ($s = 1, 2, \dots, n$). Since $\text{tr}(a^i) = \text{tr}(a^j)(i, j = 1, 2, \dots, n)$, we may write

$$AP = \begin{pmatrix} \lambda_1 + \beta & \lambda_1 & \cdots & \lambda_1 \\ \lambda_2 & \lambda_2 + \beta & \cdots & \lambda_2 \\ \cdots & \cdots & \cdots & \cdots \\ \lambda_n & \lambda_n & \cdots & \lambda_n + \beta \end{pmatrix}$$

for some $P \in \mathbf{P}_n$ and $\beta \in \mathbf{R}$. Without loss of generality, we assume $\beta > 0$. Since

$$(M_1, M_2, \dots, M_n) = (A\alpha)' \sim AP\alpha \sim AQ\alpha, \forall Q \in \mathbf{P}_n,$$

we have $M_1 = (\lambda_m + \beta)\alpha_1 + \sum_{k=2}^n \lambda_m \alpha_k, (m = 1, 2, \dots, n)$. So $(\lambda_m + \beta)\alpha_1 + \sum_{k=2}^n \lambda_m \alpha_k = (\lambda_l + \beta)\alpha_1 + \sum_{k=2}^n \lambda_l \alpha_k$, i.e. $\lambda_m = \lambda_k = \lambda$ for every $m, l = 1, 2, \dots, n$, i.e.

$$AP = \begin{pmatrix} \lambda + \beta & \lambda & \cdots & \lambda \\ \lambda & \lambda + \beta & \cdots & \lambda \\ \cdots & \cdots & \cdots & \cdots \\ \lambda & \lambda & \cdots & \lambda + \beta \end{pmatrix}$$

It is easy to verify that A is a strictly isotone.

Claim 3. Suppose that there is some row a_m of A such that every components of a_m is equal to same real number, i.e. $a_m = (\lambda_m, \lambda_m, \dots, \lambda_m)$. We claim that $a_i = (\lambda_i, \lambda_i, \dots, \lambda_i)$ for any $1 \leq i \leq n$.

If the claim is not true, then there is some row a_k of A which contains two unequal components at least in all entries of a_k . Without loss of generality we may assume that $a_i = (\lambda_i, \lambda_i, \dots, \lambda_i)$ whenever $1 \leq i \leq k$, and a_i contains two unequal components at least whenever $k+1 \leq i \leq n$, i.e.

$$A = \begin{pmatrix} a_1 \\ a_2 \\ \dots \\ a_k \\ A_1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} a_{k+1} \\ a_{k+2} \\ \dots \\ a_n \end{pmatrix},$$

where every row of A_1 contains two unequal components at least. It is easy to verify from $A\alpha \sim AP\alpha \sim AQ\alpha$ that $A_1\alpha \sim A_1P\alpha \sim A_1Q\alpha$ for every $P, Q \in \mathbf{P}_n$. We write $(A_1\alpha)^\cdot = (N_1, N_2, \dots, N_{n-k})^T$, i.e. $N_1 \geq N_2 \geq \dots \geq N_{n-k}$. For every $P \in \mathbf{P}_n$, there is some row a_l ($k+1 \leq l \leq n$) of A_1 such that $a_l P\alpha = N_1$, i.e. $P \in I_M(a_l, \alpha)$. Hence $\sum_{i=1}^{n-k} |I_M(a_{k+i}, \alpha)| \geq n!$. On the other hand, $|I_M(a_{k+i}, \alpha)| \leq (n-1)!$ by Lemma 2.4. Thus $\sum_{i=1}^{n-k} |I_M(a_{k+i}, \alpha)| \leq (n-k)(n-1)! < n!$. This is a contradiction. Hence $a_i = (\lambda_i, \lambda_i, \dots, \lambda_i)$ for any $1 \leq i \leq n$ or

$$A = \begin{pmatrix} \lambda_1 & \lambda_1 & \dots & \lambda_1 \\ \lambda_2 & \lambda_2 & \dots & \lambda_2 \\ \dots & \dots & \dots & \dots \\ \lambda_n & \lambda_n & \dots & \lambda_n \end{pmatrix}.$$

It is easy to verify that A is a strictly isotone. This completes the proof of the theorem. \square

Corollary 3.1 *Every $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in \mathbf{R}^n$ with $\alpha_1 > \alpha_2 > \dots > \alpha_n$ is a strictly all-isotone point.*

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